Received: *June 21, 1999*, Accepted: *August 13, 1999*

Convergence of derivative expansions of the renormalization group

Tim R. Morris and John F. Tighe

Department of Physics, University of Southampton,
Highfield Southampton SO17 1BJ, UK
E-mail: trmorris@hep.ph.soton.ac.uk

Abstract: We investigate the convergence of the derivative expansion of the exact renormalization group, by using it to compute the β function of scalar $\lambda\varphi^4$ theory. We show that the derivative expansion of the Polchinski flow equation converges at one loop for certain fast falling smooth cutoffs. The derivative expansion of the Legendre flow equation trivially converges at one loop, but also at two loops: slowly with sharp cutoff (as a momentum-scale expansion), and rapidly in the case of a smooth exponential cutoff. Finally, we show that the two loop contributions to certain higher derivative operators (not involved in β) have divergent momentum-scale expansions for sharp cutoff, but the smooth exponential cutoff gives convergent derivative expansions for all such operators with any number of derivatives.

Keywords: Renormalization Regularization and Renormalons, Non-perturbative Effects.

JHEP08(1999)007

Contents

1. Introduction and discussion

It is obvious that there is a need to have good analytic non-perturbative approximation methods for quantum field theory. The exact renormalization group provides a powerful framework for formulating such approximations because the calculations can be phrased directly in terms of renormalized (i.e., continuum) quantities, thus trivially preserving a crucial property of quantum field theory, namely renormalizability (equivalently the existence of a continuum limit) [1, 2]. In recent years it has been established that the derivative expansion of the effective action, taken to some finite order, provides a robust and accurate non-perturbative approximation for scalar field theory [1]–[21].

Whilst the validity of this statement currently rests mainly on empirical fact, it is proven that the derivative expansion behaves correctly in various limits. We know that the lowest order of the derivative expansion (the local potential approximation [3]) is in a sense exact in the large N limit (of, e.g., the N-vector model) [4, 5, 6]. The form of the large field behaviour is also correctly reproduced by the expansion [2, 7, 8].

Of course it is a challenging task to prove the applicability of the derivative expansion non-perturbatively and in all generality. Note in particular that this is *not* a controlled expansion in some small parameter. The approximation lies in neglecting higher powers of p/Λ, where Λ is the effective cutoff and p some typical momentum. But in the flow equations, the typical momenta that contribute are themselves of order Λ. Thus the expansion is a numerical one, and an important but difficult question to answer is whether this numerical series converges, and indeed if so, whether it converges to the exact value.

Here we establish and extend some results on the weak coupling regime, first announced in [9]. In particular we show that for the Legendre flow equation and an exponential cutoff, the derivative expansion computation for the β function of $\lambda\varphi^4$ theory[1] converges at one and two loops to the correct result.

We see then that the derivative expansion interpolates between exact results in a number of different limits, and in particular must be accurate even at only-moderately weak coupling. Viewed from this perspective, it is hardly surprising that the fully non-perturbative results tend to be so accurate [1]–[21].

On the basis of intuitive arguments and preliminary calculations, one of us suggested that the momentum expansion of the Legendre flow equation with sharp cutoff, should converge most rapidly [1, 10]. In fact, while for the Legendre equations all cutoffs trivially supply the exact answer for the β function at one loop, the two-loop sharp cutoff result converges only very slowly to the exact answer. Utilising these results we can see that the derivative expansion result for the two-loop contribution to the four-point vertex, but expanded to second or higher order in external momenta, in fact fails to converge for sharp cutoff. Actually these results again concur with the 'phenomenology', since non-perturbatively, results with sharp cutoff are consistently worse than smooth cutoff versions [8, 11].

A similar investigation for smooth cutoff shows that the derivative expansion series converges at two loops, no matter how high order the expansion in external momenta is taken.

The derivative expansion of the Polchinski flow equation fails to converge even at one loop for sharp-cutoff, power-law cutoff, exponential cutoff and many other similar forms. Remarkably however, we identify some forms of steeply falling cutoff that do result in convergence. Their existence underpins the competitive results that may be obtained with this method to $O(\partial^2)$ [13, 14, 15, 16]. (These results tend to be obtained for all cutoffs, and depend on a finite number of parameters: none for $O(\partial^0)$, three for $O(\partial^2)$ [2, 7, 9, 13, 14].)

Momentum scale expansions with sharp cutoff and derivative expansions with power-law cutoff, applied to the Legendre flow equations, share the property that reparametrization invariance is preserved [7, 9]. However, we will show that derivative expansions with power-law cutoff actually fail badly at two loops, in the sense that at some order greater than $O(\partial^4)$ (which we determine) the result is infinite because the momentum integrals fail to converge. The reason is that high order derivative expansions with power-law cutoff result in contributions that actually diverge for

[1] in four dimensions with $\varphi \leftrightarrow -\varphi$ symmetry

JHEP08(1999)007

small coupling and therefore fail even qualitatively in this regime. These conclusions are puzzling in view of the very competitive non-perturbative $O(\partial^2)$ results obtained with power-law coupling [6, 9, 11, 17].[2]

Of course it must be borne in mind that there is no real conflict here: it is quite possible that these $O(\partial^2)$ approximations provide accurate models for scalar field theory, especially since they interpolate between exact results in various limits, even if the higher orders of the derivative expansion do not result in convergent series. It is rather that convergence of the derivative expansion at low orders of the coupling *guarantees* the accuracy of high orders of the derivative expansion in this regime.

All the calculations presented in this paper have been performed directly in terms of renormalized quantities, which is one of the beauties of this approach [1, 2, 22], but we have explicitly checked that the same results are obtained with the 'traditional' approach of introducing an overall cutoff Λ_0 and allowing this to tend to infinity.

There are many possible extensions of this work. These are discussed at the end of the paper. The plan for the rest of the paper is as follows. In section 2 we consider the Wilson/Polchinski flow equation, demonstrate that the correct β function appears at one loop for any cutoff when solved exactly, and then analyse the numerical series that follows from the derivative expansion. Apart from the summary, the remaining sections are concerned with the Legendre flow equations. Section 3 treats one loop order. The derivative expansion has no effect in this case and the correct β function is always obtained. In section 4 we discuss the specific case of the sharp cutoff Legendre flow equation, calculating the two-loop β function as a convergent momentum-scale expansion. In this section we also identify the momentum-squared operator whose series fails to converge. Section 5 treats the case of a smooth exponential cutoff, demonstrating that the resulting numerical series for the two-loop β function converges rapidly to the correct result. This time considering higher powers of external momenta does not alter the conclusions: all these operators have convergent series. Finally, in section 6 we show how the power-law cutoff fails at two loops. Section 7 contains a summary and a discussion of future directions.

JHEP08(1999)007

2. Wilson/Polchinski flow equation

As Wilson showed [23] the renormalization of quantum field theories can be understood within the context of the flow of an effective action $S_\Lambda[\varphi]$ with an effective ultra-violet (UV) cutoff Λ. Polchinski's version [24] of the flow equation can be obtained from that of Wilson by substitution [7].

Firstly, let us define a modified propagator $\Delta_{\rm UV}$ such that $\Delta_{\rm UV} = C_{\rm UV}(q^2/\Lambda^2)/q^2$. $C_{\rm UV}(x)$ is an as yet unspecified function[3] which acts as an UV cutoff and hence has

[2] However, exact $O(\partial^2)$ results with exponential coupling do appear to be better still [20].

[3] x here stands for q^2/Λ^2. $C_{\rm UV}$ is a function of only this, by Lorentz invariance and dimensions.

the properties $C_{\rm UV}(0) = 1$ and $C_{\rm UV} \to 0$ (sufficiently fast) as $q \to \infty$. Similarly, we can define $\Delta_{\rm IR} = C_{\rm IR}(q^2/\Lambda^2)/q^2$, where $C_{\rm IR} = 1 - C_{\rm UV}$ behaves as an infrared cutoff. Following ref. [1] we define $K_\Lambda = d/(d\Lambda)\Delta_{\rm UV}$ and expand the flow equation for the n point function, the ϕ^n vertex of the effective action with n external momenta:

$$\frac{\partial}{\partial\Lambda} S\big(\mathbf{p}_1,\ldots,\mathbf{p}_n;\Lambda\big) = \sum_{\{I_1,I_2\}} S\big(-\mathbf{P}_1, I_1;\Lambda\big) K_\Lambda\big(P_1\big) S\big(\mathbf{P}_1, I_2;\Lambda\big) - \\ -\frac{1}{2}\int \frac{d^4q}{(2\pi)^4} K_\Lambda(q) S\big(\mathbf{q},-\mathbf{q},\mathbf{p}_1,\ldots,\mathbf{p}_n;\Lambda\big)\,, \qquad (2.1)$$

where I_1 and I_2 are disjoint subsets of external momenta such that $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = \{\mathbf{p}_1,\ldots,\mathbf{p}_n\}$. The sum over $\{I_1, I_2\}$ utilises the Bose symmetry so pairs are counted only once, i.e., $\{I_1, I_2\} = \{I_2, I_1\}$. The momentum $\mathbf{P}_1$ is defined to be $\mathbf{P}_1 = \sum_{\mathbf{p}_i \epsilon I_1} \mathbf{p}_i$.

We first consider the four point vertex exactly (i.e., without a derivative expansion) and obtain the exact β function. We define the renormalized coupling λ to be the four point vertex at zero momentum. The only contribution comes from the tree-level six-point function with two legs tied together to give the diagram in figure 1.

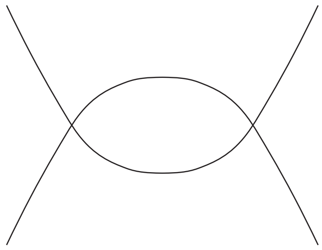

Figure 1: Feynman diagram contributing to four point function at one loop.

The tree-level six-point vertex follows from integrating up (2.1) with λ as the four-vertex. Substituting back into (2.1) we thus have (compare [1, 22])

$$\frac{\partial}{\partial\Lambda}\lambda = 3\lambda^2 \int \frac{d^4q}{(2\pi)^4} K_\Lambda(q) \left[\int_\Lambda^\infty d\Lambda_1 K_{\Lambda_1}(q)\right] = \qquad (2.2)$$

$$= 3\lambda^2 \int \frac{d^4q}{(2\pi)^4} K_\Lambda(q) \Delta_{\rm IR}\big(q^2/\Lambda^2\big) = \qquad (2.3)$$

$$= \frac{6\lambda^2}{(4\pi)^2}\frac{1}{\Lambda}\int_0^\infty dx\, C'_{\rm IR}(x) C_{\rm IR}(x) = \qquad (2.4)$$

$$= \frac{3\lambda^2}{(4\pi)^2}\frac{1}{\Lambda}\Big[C^2_{\rm IR}(\infty) - C^2_{\rm IR}(0)\Big]\,, \qquad (2.5)$$

where in (2.4) the prime means differentiation with respect to x and in (2.2) the tree-level six-point function provides the Λ_1 integral. Since the Λ_1 integral is UV convergent we do not need an overall cutoff (Λ_0), but instead go direct to the continuum limit ($\Lambda_0 = \infty$). Using the definition of $C_{\rm IR} = 1 - C_{\rm UV}$, we see that the β function ($\beta(\Lambda) \equiv \Lambda(\partial/\partial\Lambda)\lambda$) is correctly [25] reproduced as $3\lambda^2/(4\pi)^2$ at one loop. (We remind the reader that the one and two loop β function coefficients are universal for massless four-dimensional $\lambda\phi^4$ theory.)

Now consider evaluating the effective action in terms of a derivative expansion. This means that the $\Delta_{\rm IR}$ in (2.3), which is the momentum dependent part of the classical six-point contribution, must be expanded in q. The intermediate result (2.4) is thus replaced by[4]

$$\beta = \frac{6\lambda^2}{(4\pi)^2}\sum_{n=1}^{\infty}\frac{C_{\rm UV}^{(n)}(0)}{n!}\int_0^{\infty} dx\, x^n C'_{\rm UV}(x)\,, \tag{2.6}$$

where $C_{\rm UV}^{(n)}$ is the n-th x-derivative of $C_{\rm UV}(x)$. Taking the specific example of sharp cutoff $C_{\rm UV} = \theta(1-x)$ we trivially obtain convergence to the wrong answer, namely zero, since $C_{\rm UV}^{(n)}(0) = 0$ for all $n \geq 1$. However for sharp cutoff, the derivative expansion actually does not exist, and a careful determination of the sharp cutoff limit leads one to the Legendre equations even in the exact case [10]. Therefore we will consider the sharp cutoff from now on only in the Legendre flow equations. If we choose a cutoff that decays as a power of q^2, it is clear that at some n the integral in (2.6) diverges. In order to avoid this problem we must make the cutoff decay faster than a power. If we choose an exponential cutoff of the form $C_{\rm UV} = e^{-q^2/\Lambda^2}$ however, we still have problems:

$$\beta = \frac{6\lambda^2}{(4\pi)^2}\sum_{n=1}^{\infty}(-1)^{n+1}\,. \tag{2.7}$$

Clearly this does not converge. In fact, the reader can readily confirm that exponential decay with any power ($C_{\rm UV} = e^{-x^m}$) still fails to give a convergent derivative expansion. The $\sim 1/n!$ behaviour of the Taylor expansion coefficients is not enough in these cases to overcome the $\sim n!$ behaviour of the integrals. Clearly any cutoff that behaves as $\sim e^{-x^m}$ for large x will thus also fail. Indeed if these cutoffs are not entire as functions of complex x, the Taylor expansion coefficients will decay only as a power $\sim 1/R^n$ (R the distance from the origin to the nearest singularity) and will result in badly divergent numerical series.

However, we can obtain convergence if we insist that the UV cutoff is entire but falls fast enough as $x \to \infty$ to ensure that the integrals in (2.6) grow much slower than n![5] With $C_{\rm UV}(x) = \exp(1-e^x)$ for example, we obtain

$$\beta = \frac{3\lambda^2}{(4\pi)^2}\Big\{1.193 + 0 - 0.194 - 0.060 + 0.032 + \cdots\Big\}, \tag{2.8}$$

with the series in braces summing to 1.000286 after twelve terms. With $C_{\rm UV} = \exp\{e - \exp(e^x)\}$ we obtain

$$\beta = \frac{3\lambda^2}{(4\pi)^2}\Big\{1.278 - 0.164 - 0.130 - 0.014 + 0.019 + \cdots\Big\}, \tag{2.9}$$

[4]See also [16].

[5]In the limit of sharp cutoff the integrals are all 1.

which also converges, with the series in braces summing to 0.999551 after twelve terms. (The integrals of (2.6) were calculated numerically.)

It would be very interesting to investigate if these forms of cutoff yield convergent derivative expansions at higher loops also. However, in this paper we now turn to the Legendre flow equations: these have inherently better convergence properties at the very least because, being 1PI (one particle irreducible), there are no tree-level corrections and thus numerical series arising from derivative expansion do not appear until the two-loop level.

3. Legendre flow equation at one loop

Following the notation of ref. [1], see also [18, 19], we have the Legendre flow equation for a general cutoff (with the vacuum energy dropped):

$$\frac{\partial}{\partial\Lambda}\Gamma[\varphi^c] = -\frac{1}{2}\mathbf{tr}\left\{\frac{K_\Lambda}{(1+\Delta_{\mathrm{IR}}\Sigma)^2}\cdot\hat{\Gamma}\cdot\left(1+\left[\Delta_{\mathrm{IR}}^{-1}+\Sigma\right]^{-1}\cdot\hat{\Gamma}\right)^{-1}\right\}, \tag{3.1}$$

where $\Gamma[\varphi^c]$ is the generator of 1PI Greens functions, φ^c is the classical field, $\hat{\Gamma}[\varphi^c]$ the second field differential less Σ, the field independent part, i.e., the effective self energy. Equation (3.1) can be expanded in terms of φ^c to give the expanded Legendre flow equation although care is required when taking the sharp cutoff limit [1, 10]. However at one loop (irrespective of exact form of cutoff), the only contribution to the flow of the four point function will arise from

$$\begin{aligned}\frac{\partial}{\partial\Lambda}\Gamma\Big(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4;\Lambda\Big) &= \\ = \int\frac{d^4q}{(2\pi)^4}K_\Lambda(q)\sum_{\{I_1,I_2\}}\Gamma\Big(\mathbf{q},-\mathbf{q}-\mathbf{P}_1,I_1;\Lambda\Big)&\Delta_{\mathrm{IR}}\left(|\mathbf{q}+\mathbf{P}_1|\right)\Gamma\Big(\mathbf{q}-\mathbf{P}_2,-\mathbf{q},I_2;\Lambda\Big),\end{aligned} \tag{3.2}$$

where the notation is the same as used in (2.1). With λ defined as the four-point 1PI vertex at zero momenta and substituting λ—the tree-level result, in the right-hand side, this reduces to

$$\frac{\partial}{\partial\Lambda}\lambda = 3\lambda^2\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^4}\left(\frac{d}{d\Lambda}C_{\mathrm{IR}}\left(q^2/\Lambda^2\right)\right)C_{\mathrm{IR}}\left(q^2/\Lambda^2\right). \tag{3.3}$$

Derivative expansion corresponds to an expansion in the external momenta of Γ's vertices, but in this case the relevant vertices have no external momentum dependence. It is easy to see that (3.3) is the same as (2.3) and thus for the Legendre equations, even in the derivative expansion, we obtain the exact one-loop β function $3\lambda^2/(4\pi)^2$ irrespective of the exact form of cutoff function.

4. Sharp cutoff at two loops

With due caution the sharp cutoff limit $[C_{\rm UV} = \theta(\Lambda - q)]$ of (3.1) is taken [1], and after expanding in φ^c we obtain the sharp cutoff expanded Legendre flow equation for n external momenta:

$$\frac{\partial}{\partial\Lambda}\Gamma\Big(\mathbf{p}_1,\dots,\mathbf{p}_n;\Lambda\Big) = \int\frac{d^4q}{(2\pi)^4}\frac{\delta(q-\Lambda)}{q^2+\Sigma(q;\Lambda)}E\Big(\mathbf{q},\mathbf{p}_1,\dots,\mathbf{p}_n;\Lambda\Big)\,, \tag{4.1}$$

where

$$\begin{aligned} &E\Big(\mathbf{q},\mathbf{p}_1,\dots,\mathbf{p}_n;\Lambda\Big) = \\ &\quad = -\frac{1}{2}\Gamma\Big(\mathbf{q},-\mathbf{q},\mathbf{p}_1,\dots,\mathbf{p}_n;\Lambda\Big) + \\ &\qquad + \sum_{\{I_1,I_2\}}\Gamma\Big(\mathbf{q},-\mathbf{q}-\mathbf{P}_1,I_1;\Lambda\Big)G\Big(|\mathbf{q}+\mathbf{P}_1|;\Lambda\Big)\Gamma\Big(\mathbf{q}-\mathbf{P}_2,-\mathbf{q},I_2;\Lambda\Big) - \\ &\qquad - \sum_{\{I_1,I_2\},I_3}\Gamma\Big(\mathbf{q},-\mathbf{q}-\mathbf{P}_1,I_1;\Lambda\Big)G\Big(|\mathbf{q}+\mathbf{P}_1|;\Lambda\Big)\times \\ &\qquad\qquad \times\Gamma\Big(\mathbf{q}+\mathbf{P}_1,-\mathbf{q}+\mathbf{P}_2,I_3;\Lambda\Big)G\Big(|\mathbf{q}-\mathbf{P}_2|;\Lambda\Big)\Gamma\Big(\mathbf{q}-\mathbf{P}_2,-\mathbf{q},I_2;\Lambda\Big) + \cdots\,. \end{aligned} \tag{4.2}$$

Similarly to before, $\mathbf{P}_i = \sum_{\mathbf{p}_j\in I_i}\mathbf{p}_j$ and $\sum_{\{I_1,I_2\},I_3,\dots,I_m}$ is a sum over disjoint subsets $I_i \cap I_j = \emptyset$ $(\forall i,j)$ with $\bigcup_{i=1}^m I_i = \{\mathbf{p}_1,\dots,\mathbf{p}_n\}$. Again, the symmetrization $\{I_1, I_2\}$ means this pair is counted only once. $G(p;\Lambda)$ is defined by

$$G(p;\Lambda) \equiv \frac{\theta(p-\Lambda)}{p^2+\Sigma(p;\Lambda)}\,, \tag{4.3}$$

where Σ is again the (field independent) self energy.

By iteration we can now solve (4.1) to two-loop order. We split the four point function into two parts, momentum free $[\lambda(\Lambda)]$ and momentum dependent [22]:

$$\Gamma\Big(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4;\Lambda\Big) = \lambda(\Lambda) + \gamma\Big(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4;\Lambda\Big)\,, \tag{4.4}$$

where

$$\gamma\Big(0,0,0,0;\Lambda\Big) = 0\,. \tag{4.5}$$

Therefore at one loop

$$
\begin{aligned}
&\gamma(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4;\Lambda) = \\
&= -\lambda^2 \int_\Lambda^\infty d\Lambda_1 \int \frac{d^4q}{(2\pi)^4}\frac{\delta(q-\Lambda_1)}{q^2}\sum_{i=2}^{4}\left\{\frac{\theta(|\mathbf{q}+\mathcal{P}_i|-\Lambda_1)}{(\mathbf{q}+\mathcal{P}_i)^2}-\frac{\theta(q-\Lambda_1)}{q^2}\right\} \qquad (4.6)\\
&= -\frac{\lambda^2}{4\pi^3}\sum_{i=2}^{4}\int_\Lambda^\infty \frac{dq}{q}\left\{-\frac{1}{2}+\int_{-1}^{1}dx\;\theta(2x+\mathcal{P}_i/q)\frac{\sqrt{1-x^2}}{1+2x\mathcal{P}_i/q+\mathcal{P}_i^2/q^2}\right\} \qquad (4.7)\\
&= \frac{\lambda^2}{4\pi^3}\sum_{i=2}^{4}\left\{\frac{1}{6}\frac{\mathcal{P}_i}{\Lambda}+\frac{1}{720}\left(\frac{\mathcal{P}_i}{\Lambda}\right)^3+\frac{3}{44800}\left(\frac{\mathcal{P}_i}{\Lambda}\right)^5+\cdots\right\}, \qquad (4.8)
\end{aligned}
$$

where $\mathcal{P}_\mathbf{i} = \mathbf{p}_1 + \mathbf{p}_i$ and $x = \mathcal{P}_\mathbf{i}\cdot\mathbf{q}/\mathcal{P}_i$. Note that the subtraction of the momentum independent part in (4.6) ensures that the Λ_1 integral converges, allowing (again) the upper limit to be set as ∞. In (4.7) we perform the Λ_1 integral, noting that effectively here $\theta(0) = 1/2$ [1]. By absorbing the step function into the x limit, the term in braces may be expanded in momentum-scale $\mathcal{P}_i = |\mathcal{P}_\mathbf{i}|$ [1, 10] to give (4.8). Alternatively the step function may be expanded directly [1]

$$
\theta\left(\frac{\mathcal{P}_i}{2q}+x\right) = \theta(x) + \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{\mathcal{P}_i}{2q}\right)^n \delta^{(n-1)}(x)\,, \qquad (4.9)
$$

where $\delta^{(n-1)}(x)$ is the $(n-1)$th derivative of $\delta(x)$ with respect to x. Of course (4.8) agrees with the bare version already computed in ref. [1].

We will also need the renormalized one-loop self-energy which we obtain directly from the flow equation as follows:

$$
\frac{\partial}{\partial\Lambda}\Sigma(p;\Lambda) = -\frac{\lambda}{2}\int\frac{d^4q}{(2\pi)^4}\frac{\delta(q-\Lambda)}{q^2} = -\frac{\lambda}{(4\pi)^2}\Lambda\,. \qquad (4.10)
$$

Integrating up (4.10) we must not introduce a mass scale as this is a massless theory. Consequently, the uniquely determined self energy has to be

$$
\Sigma(p;\Lambda) = -\frac{\lambda}{(4\pi)^2}\frac{\Lambda^2}{2} \qquad (4.11)
$$

(with or without momentum-scale expansion).

At two loops, the diagrams in (b) and (c) of figure 2 will contribute to the β function. Diagram (a) might be expected to be included, however on setting external momenta to zero the iterand of this topology vanishes by (4.5). The reason is that in calculating renormalized quantities directly, (a) is already incorporated in the one-loop running $\lambda(\Lambda)$. (In the more traditional calculation, (a) only has a divergent part which is then removed on renormalization.)

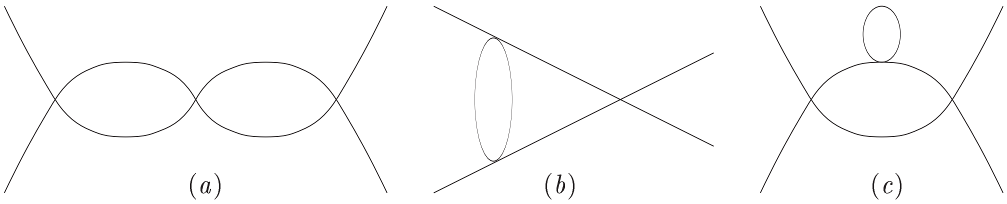

Figure 2: Feynman diagrams contributing to the four point function at two loops.

The flow equation at this order is

$$\begin{aligned}\frac{\partial}{\partial\Lambda}\lambda(\Lambda) =&\frac{1}{\Lambda}\frac{3\lambda^2}{(4\pi)^2}+3\lambda^3\int\frac{d^4q}{(2\pi)^4}\frac{\delta(q-\Lambda)}{q^2}\left[\frac{\Lambda^2}{(4\pi)^2}\frac{\theta(q-\Lambda)}{q^4}\right.\\&-\frac{1}{2}\int_\Lambda^\infty d\Lambda_1\int\frac{d^4p}{(2\pi)^4}\frac{\delta(p-\Lambda_1)}{p^2}\left\{\frac{3\theta^2(p-\Lambda_1)}{p^4}+\right.\\&+\frac{4\theta^2(|\mathbf{p}+\mathbf{q}|-\Lambda_1)}{|\mathbf{p}+\mathbf{q}|^4}+\frac{8\theta(|\mathbf{p}+\mathbf{q}|-\Lambda_1)\theta(p-\Lambda_1)}{p^2|\mathbf{p}+\mathbf{q}|^2}+\\&\left.\left.+\frac{4\theta(q-\Lambda)}{q^2}\left(\frac{\theta(|\mathbf{p}+\mathbf{q}|-\Lambda_1)}{|\mathbf{p}+\mathbf{q}|^2}-\frac{\theta(p-\Lambda_1)}{p^2}\right)\right\}\right].\end{aligned}\tag{4.12}$$

In here, the first $O(\lambda^3)$ term arises from the expansion of the self energy (4.11), i.e., topology figure 2c. The next also gives rise to topology (c), but is now generated by the one-loop six-point vertex with two legs tied together. The six-point vertex also provides the next two terms in the form of figure 2b. The final line arises from iterating the four-point function of (4.8) in the flow equation.

If we consider the first two (self energy) terms we find they evaluate to $3(\lambda^3/(4\pi)^4)(1/\Lambda)$ and $-3(\lambda^3/(4\pi)^4)(1/\Lambda)$ respectively ($\theta^2(0)\equiv 1/3$ here [1]), and thus cancel. Actually, it may be shown that these self energy terms cancel for any cutoff. Neither is affected by momentum-scale expansion, since no external momentum dependence is involved. In the next two contributions the momentum expansion does affect the embedded one-loop terms and corresponds in this case to an expansion in q/p. Proceeding as before, we obtain the expansion of the first of these to be

$$-12\frac{\lambda^3}{(4\pi)^4}\frac{1}{\Lambda}\frac{1}{\pi}\left(\frac{\pi}{2}-\frac{10}{9}+\frac{\pi}{4}-\frac{63}{100}+\frac{\pi}{6}-\frac{7035}{15680}+\cdots\right)\tag{4.13}$$

and of the other to be

$$-12\frac{\lambda^3}{(4\pi)^4}\frac{1}{\Lambda}\frac{1}{\pi}\left(\frac{\pi}{2}-\frac{2}{9}-\frac{1}{300}-\frac{3}{15680}+\cdots\right).\tag{4.14}$$

We see that the second series converges rapidly, but the first series is very slowly converging. We are sure that it does actually converge because we computed the first 80 terms. These continue to oscillate and the partial sums show clear signs of slow convergence, with 79 terms giving $S_{79} = .918$, and 80 terms $S_{80} = .879$ for the bracketed series in (4.13). The average of successive partial sums $(1/2)(S_{2n} + S_{2n-1})$ fits a form $a + b/n^2$ well for large n and on this basis we estimate $S_\infty = .89828$ to 5dp. The final part of (4.12) reproduces the previously published value [1] of

$$\frac{\lambda^3}{(4\pi)^4}\frac{1}{\Lambda}\frac{1}{\pi}\left(8 + \frac{1}{15} + \frac{9}{2800} + \cdots\right). \tag{4.15}$$

At this level of perturbation theory wave function renormalization appears through $\Sigma(k;\Lambda)|_{O(k^2)} = [Z(\Lambda) - 1]k^2$ arising from figure 3,

$$\begin{aligned}
&k^2\frac{\partial}{\partial\Lambda}Z(\Lambda) = \\
&= \lambda^2 \int \frac{d^4q}{(2\pi)^4}\frac{\delta(q-\Lambda)}{q^2}\int_\Lambda^\infty d\Lambda_1 \int \frac{d^4p}{(2\pi)^4}\frac{\delta(p-\Lambda_1)}{p^2}\frac{\theta(|\mathbf{p}+\mathbf{q}+\mathbf{k}| - \Lambda_1)}{|\mathbf{p}+\mathbf{q}+\mathbf{k}|^2}\Bigg|_{O(k^2)} = \\
&= -\frac{\lambda^2}{4\pi^3}\int \frac{d^4q}{(2\pi)^4}\frac{\delta(q-\Lambda)}{q^2}\left\{\frac{1}{6}\frac{|\mathbf{q}+\mathbf{k}|}{\Lambda} + \frac{1}{720}\frac{|\mathbf{q}+\mathbf{k}|^3}{\Lambda^3} + \frac{3}{44800}\frac{|\mathbf{q}+\mathbf{k}|^5}{\Lambda^5} + \cdots\right\}\Bigg|_{O(k^2)} = \\
&= -\frac{\lambda^2 k^2}{(4\pi)^4}\frac{1}{\Lambda}\frac{1}{\pi}\left(\frac{1}{2} + \frac{1}{48} + \frac{3}{1280} + \cdots\right).
\end{aligned} \tag{4.16}$$

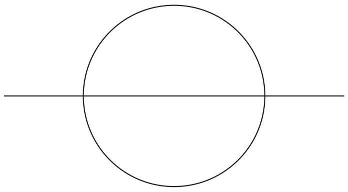

Figure 3: Feynman diagram contributing to wave function renormalization at two loops.

Note that the second line is the expanded one-loop four-point vertex (4.8). The net effect of expanding to second order in k and averaging over the angles, is to convert $|\mathbf{q}+\mathbf{k}|^n$ into $(1/8)n(n+2)q^{n-2}k^2$.

As a consequence, the properly normalised definition of the β function is now $\beta(\Lambda)=\Lambda(\partial/\partial\Lambda)\,(\lambda(\Lambda)/Z^2)$. Collecting all terms, the β function at two loops for the Legendre flow equation with sharp cutoff is found to be

$$\begin{aligned}
\beta(\Lambda) &= \Lambda\frac{\partial}{\partial\Lambda}\lambda - 2\lambda\Lambda\frac{\partial}{\partial\Lambda}Z(\Lambda) = \\
&= 3\frac{\lambda^2}{(4\pi)^2} - \frac{\lambda^3}{(4\pi)^4}\frac{1}{\pi}\left\{12\left(\frac{\pi}{2} - \frac{10}{9} + \frac{\pi}{4} - \frac{63}{100} + \frac{\pi}{6} - \frac{7035}{15680} + \cdots\right) + \right. \\
&\quad + 12\left(\frac{\pi}{2} - \frac{2}{9} - \frac{1}{300} - \frac{3}{15680} + \cdots\right) - \left(8 + \frac{1}{15} + \frac{9}{2800} + \cdots\right) - \\
&\quad \left. - \left(1 + \frac{1}{24} + \frac{3}{640} + \cdots\right)\right\},
\end{aligned} \tag{4.17}$$

where in the first line we use $Z(\Lambda) = 1 + O(\lambda^2)$. Using our above result for the slow series, we see that the rest rapidly converges towards the exact expression [25]

$$\beta(\Lambda) = 3\frac{\lambda^2}{(4\pi)^2} - \frac{17}{3}\frac{\lambda^3}{(4\pi)^4}. \tag{4.18}$$

Our observation below (4.16) however, shows that for second order in the external momentum, equivalently $O(\partial^2)$ operators, the n^{th} coefficient of momentum-scale expansion is effectively multiplied by $\sim n^2$. For an $O(\partial^{2r})$ operator, the q^n term is that of the derivative-free operator (i.e., with the same number of fields) with q^n replaced by $\sim \binom{n/2}{2r} q^{n-2r} k^{2r}$ which yields a multiplier $\sim n^{2r}$ for large n. Unless the coefficients in the numerical series fall faster than a power of n (e.g., $\sim 1/n!$ or $1/R^n$, $R > 1$), we will be able to find a sufficiently high derivative operator whose derivative expansion series fails to converge. While (probably) all but one of the sharp-cutoff series do have coefficients that fall faster than a power of n, the slow series (4.13) barely manages to converge at all. Thus it is clear without calculation, that the contribution figure 2b taken to second or higher order in its external momenta, i.e., in particular the $O(k^{2r})$ $r \geq 1$ coefficients of

$$-6\lambda^3 \int \frac{d^4q}{(2\pi)^4} \frac{\delta(q-\Lambda)}{q^2} \int_\Lambda^\infty d\Lambda_1 \int \frac{d^4p}{(2\pi)^4} \frac{\delta(p-\Lambda_1)}{p^2} \frac{\theta^2(|\mathbf{p}+\mathbf{q}+\mathbf{k}|-\Lambda_1)}{|\mathbf{p}+\mathbf{q}+\mathbf{k}|^4}, \tag{4.19}$$

yield momentum-scale expansion series that fail to converge.

5. Smooth cutoff

We return to smooth cutoffs, this time within the context of the Legendre flow equation. Equation (3.1) can now be expanded in φ^c directly without the difficulties that arise from taking the sharp limit [1]. Thus we have:

$$\frac{\partial}{\partial\Lambda}\Gamma\Big(\mathbf{p}_1,\ldots,\mathbf{p}_n;\Lambda\Big) = \int \frac{d^4q}{(2\pi)^4} \frac{q^2(\partial/\partial\Lambda)C_{\mathrm{UV}}(q^2/\Lambda^2)}{\big[q^2 + C_{\mathrm{IR}}(q^2/\Lambda^2)\Sigma(q;\Lambda)\big]^2} E\Big(\mathbf{q},\mathbf{p}_1,\ldots,\mathbf{p}_n;\Lambda\Big), \tag{5.1}$$

where $E(\mathbf{q},\mathbf{p}_1,\ldots,\mathbf{p}_n;\Lambda)$ is as defined in (4.2) except now $G(p;\Lambda)$ is defined by

$$G(p;\Lambda) \equiv \frac{C_{\mathrm{IR}}(q^2/\Lambda^2)}{q^2 + C_{\mathrm{IR}}(q^2/\Lambda^2)\Sigma(q;\Lambda)}. \tag{5.2}$$

Concentrating on the exponential cutoff, $C_{\mathrm{UV}} = e^{-q^2/\Lambda^2}$, at one loop we find the renormalized four-point function is given by [10]

$$\gamma\Big(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4;\Lambda\Big) =$$

$$= -\lambda^2 \sum_{i=2}^{4} \int_\Lambda^\infty d\Lambda_1 \int \frac{d^4q}{(2\pi)^4} \left(\frac{\partial}{\partial\Lambda_1}\frac{e^{-q^2/{\Lambda_1}^2}}{q^2}\right) \left\{ \frac{\big(1-e^{-|\mathbf{q}+\mathcal{P}_\mathbf{i}|^2/{\Lambda_1}^2}\big)}{|\mathbf{q}+\mathcal{P}_\mathbf{i}|^2} - \frac{\big(1-e^{-q^2/{\Lambda_1}^2}\big)}{q^2} \right\} \tag{5.3}$$

$$= -2\frac{\lambda^2}{(4\pi)^2} \sum_{i=2}^{4} \int_\Lambda^\infty \frac{d\Lambda_1}{\Lambda_1} \left\{ \left(\frac{\Lambda_1}{\mathcal{P}_i}\right)^2 \left(1 - e^{-{\mathcal{P}_i}^2/2{\Lambda_1}^2}\right) - \frac{1}{2} \right\} = \tag{5.4}$$

$$= -\frac{\lambda^2}{2(4\pi)^2} \sum_{i=2}^{4} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!n} \left(\frac{{\mathcal{P}_i}^2}{2{\Lambda_1}^2}\right)^n. \tag{5.5}$$

The expression in (5.4) can be obtained from (5.3) either by expanding the exponentials, performing the integration over momentum space and then resumming, or by using

$$\frac{\left(1-e^{-q^2/\Lambda^2}\right)}{q^2}=\frac{1}{\Lambda^2}\int_0^1 da\; e^{-aq^2/\Lambda^2} \tag{5.6}$$

and interchanging the order of integration.

We again need to consider the one-loop self energy and once more calculate its differential to be

$$\frac{\partial}{\partial\Lambda}\Sigma(p;\Lambda)=-\frac{\lambda}{2}\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2}\frac{\partial}{(\partial\Lambda)}\left(e^{-q^2/\Lambda^2}\right)=-\frac{\lambda}{(4\pi)^2}\Lambda \tag{5.7}$$

which integrates uniquely to

$$\Sigma(p;\Lambda)=-\frac{\lambda}{(4\pi)^2}\frac{\Lambda^2}{2}, \tag{5.8}$$

by virtue of this being a massless theory. While this is exactly the same value as obtained with a sharp cutoff, it can be shown that this is a coincidence. To two-loop order, the Legendre flow equation for a smooth cutoff is

$$\begin{aligned}
\frac{\partial}{\partial\Lambda}\lambda(\Lambda)=&\frac{1}{\Lambda}\frac{3\lambda^2}{(4\pi)^2}+3\lambda^3\int\frac{d^4q}{(2\pi)^4}\frac{\partial/(\partial\Lambda)\left(e^{-q^2/\Lambda^2}\right)}{q^2}\left[\frac{3\Lambda^2}{2(4\pi)^2}\frac{\left(1-e^{-q^2/\Lambda^2}\right)^2}{q^4}-\right.\\
&-\frac{1}{2}\int_\Lambda^\infty d\Lambda_1\int\frac{d^4p}{(2\pi)^4}\frac{\partial/(\partial\Lambda_1)\left(e^{-p^2/\Lambda_1^2}\right)}{p^2}\left\{3\frac{\left(1-e^{-p^2/\Lambda_1^2}\right)^2}{p^4}\times\right.\\
&\times\frac{4\left(1-e^{-|\mathbf{p}+\mathbf{q}|^2/\Lambda_1^2}\right)^2}{|\mathbf{p}+\mathbf{q}|^4}+\frac{8\left(1-e^{-|\mathbf{p}+\mathbf{q}|^2/\Lambda_1^2}\right)\left(1-e^{-p^2/\Lambda_1^2}\right)}{p^2|\mathbf{p}+\mathbf{q}|^2}+\\
&\left.\left.+\frac{4\left(1-e^{-q^2/\Lambda^2}\right)}{q^2}\left(\frac{\left(1-e^{-|\mathbf{p}+\mathbf{q}|^2/{\Lambda_1}^2}\right)}{|\mathbf{p}+\mathbf{q}|^2}-\frac{\left(1-e^{-p^2/\Lambda_1^2}\right)}{p^2}\right)\right\}\right].
\end{aligned} \tag{5.9}$$

The terms in (5.9) are in direct correspondence to those of (4.12). Performing the required integrals we find the diagram arising from the insertion of the self energy into the one loop four point function to be $9\ln(4/3)(\lambda^3/(4\pi)^4)(1/\Lambda)$, while the self energy diagram coming from the six-point vertex with two legs joined is $-9\ln(4/3)\lambda^3(4\pi)^4(1/\Lambda)$. Again they cancel, as required.

The next three contributions are all of the form figure $2b$. The first of these gives

$$12\frac{\lambda^3}{(4\pi)^4}\frac{1}{\Lambda}\left(\ln\frac{4}{3}+\sum_{n=2}^{\infty}(-1)^n\left[\ln\frac{4}{3}-\frac{1}{n}\sum_{s=2}^{n}\binom{n}{s}\frac{(-1)^s}{s-1}\left\{1-\frac{1}{2^{s-2}}+\frac{1}{3^{s-1}}\right\}\right]\right), \tag{5.10}$$

(when expanded) which numerically sums to $(\lambda^3/(4\pi)^4)(1/\Lambda)(-2.45411725)$, and the second

$$-24\frac{\lambda^3}{(4\pi)^4}\frac{1}{\Lambda}\left(\ln\frac{4}{3}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+1)}\left\{\left(\frac{2}{3}\right)^n-\left(\frac{1}{2}\right)^n\right\}\right), \tag{5.11}$$

which sums exactly to $12(\lambda^3/(4\pi)^4)(1/\Lambda)[9\ln 3-2\ln 2-5\ln 5]$. The final part of figure $2b$, arising from the iterated value of $\gamma(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4;\Lambda)$, gives [10]

$$-12\frac{\lambda^3}{(4\pi)^4}\frac{1}{\Lambda}\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+1)}\frac{1}{2^n}\left(1-\frac{1}{2^{n+1}}\right). \tag{5.12}$$

This can be shown to sum exactly to $6(\lambda^3/(4\pi)^4)(1/\Lambda)[6\ln 3+4\ln 2-5\ln 5-1]$.

For wave function renormalization we have, similarly to before,

$$\begin{aligned}k^2\frac{\partial}{(\partial\Lambda)}Z(\Lambda) = -\lambda^2\int\frac{d^4q}{(2\pi)^4}\left(\frac{\partial}{(\partial\Lambda)}\frac{e^{-q^2/\Lambda^2}}{q^2}\right)\times\\ \times\int_{\Lambda}^{\infty}d\Lambda_1\int\frac{d^4p}{(2\pi)^4}\left(\frac{\partial}{\partial\Lambda}\frac{e^{-p^2/\Lambda_1{}^2}}{p^2}\right)\left(\frac{1-e^{-|\mathbf{p}+\mathbf{q}+\mathbf{k}|^2/\Lambda_1{}^2}}{|\mathbf{p}+\mathbf{q}+\mathbf{k}|^2}\right)\Bigg|_{O(k^2)} =\\ =\frac{\lambda^2k^2}{(4\pi)^4}\frac{1}{\Lambda}\sum_{n=2}^{\infty}\frac{(-1)^n}{2^n},\end{aligned} \tag{5.13}$$

which equates to $1/6(\lambda^2k^2/(4\pi)^4)(1/\Lambda)$. Hence the β function to two loops is given as

$$\begin{aligned}\beta(\Lambda) =& 3\frac{\lambda^2}{(4\pi)^2}-\frac{12\lambda^3}{(4\pi)^4}\Bigg\{\ln\frac{4}{3}+\\ &+\sum_{n=1}^{\infty}(-1)^n\Bigg[\ln\frac{4}{3}-\frac{1}{12}\left(\frac{1}{2}\right)^n+\frac{1}{n(n+1)}\left(2\left(\frac{2}{3}\right)^n-\frac{1}{2^n}-\frac{1}{2^{2n+1}}\right)-\\ &-\frac{1}{n+1}\sum_{s=2}^{n+1}\binom{n+1}{s}\frac{(-1)^s}{s-1}\left\{1-\frac{1}{2^{s-2}}+\frac{1}{3^{s-1}}\right\}\Bigg]\Bigg\} =\\ =& 3\frac{\lambda^2}{(4\pi)^2}-\frac{\lambda^3}{(4\pi)^4}\left[72\ln 3-48\ln 2-30\ln 5+2.45411725+6-\frac{1}{3}\right],\end{aligned} \tag{5.14}$$

which gives the expected form of (4.18).

We see that for smooth exponential cutoff, the derivative expansion series all have coefficients that fall at least as fast as $1/R^n$, with $R > 1$. (It may be shown that the n^{th} term in (5.10) falls faster than $(1/n)(2/3)^n$ for large n.) It follows then, from the discussion at the end of the previous section, that the diagrams of figure 2 and figure 3 expanded to the $(2r)^{\text{th}}$ power of the external momenta, still yield convergent derivative expansion series in this case. In fact the situation is even better than this. As before, the k^{2r} term (k some external momentum) effectively converts the power q^{2n} in the expanded terms of (5.9) to $\sim n^{2r}k^{2r}q^{2n-2r}$, but here the coefficients of q^{2n} go like $\sim 1/n!$ or better and the power of q is integrated against e^{-q^2/Λ^2}. Converting q^{2n} to q^{2n-2r} means that the $1/n!$ is now incompletely cancelled by the q integral, leaving a remainder $\sim 1/n^{2r}$ for large n. Thus the net result is that not only do the series for all the higher derivative operators converge, they converge just as fast as the zero external momentum diagrams!

6. Power-law cutoff

The final cutoff we consider is that of a power-law, i.e., $C_{\text{UV}}(q,\Lambda) = 1/[1+(q/\Lambda)^{2\kappa+2}]$ where κ is a non-negative integer. Recall that this leads to a well defined derivative expansion to all orders if $\kappa > D/2 - 1$ [7] (in the sense that all the momentum integrals converge, D being the space-time dimension). Despite this, problems arise with the integrals at two loops, when we consider perturbation theory. For example, consider the integral pertaining to figure $2b$, obtained by the iteration of the four-point function at one loop. We have the following contribution to two-loop β function:

$$\begin{aligned}\sim \frac{\lambda^3}{\Lambda^{2\kappa+3}} \int d^4q \frac{q^{4\kappa}}{[1+(q/\Lambda)^{2\kappa+2}]^3} \int_\Lambda^\infty \frac{d\Lambda_1}{\Lambda_1^{2\kappa+3}} \int d^4p \, \times \\ \times \frac{p^{2\kappa}}{[1+(p/\Lambda_1)^{2\kappa+2}]^2} \left[1 - \frac{1}{1+(|\mathbf{q}+\mathbf{p}|/\Lambda_1)^{2\kappa+2}}\right] \frac{1}{|\mathbf{q}+\mathbf{p}|^2}\,. \end{aligned} \tag{6.1}$$

Derivative expansion requires the inner integral (the one-loop four-point function) to be expanded in powers of its external momentum q. The expansion of one-loop terms always exists to all orders but, once the power q^{2m} is such that $m \geq \kappa + 1$ (or $\kappa + 3 - D/2$ in general space-time dimension), the second loop integral over q, fails to converge. In this case, even the *coefficients* of the derivative expansion series are infinite. In the worst case, $\kappa = 2$, this happens at $O(\partial^6)$.

At first sight there is a conflict with our earlier statement, that the integrals converge if $\kappa > D/2 - 1$. The resolution is that, non-perturbatively there exists also a factor $\sim 1/[q^2 + C_{\text{IR}}\Sigma]^3$ from (5.1) and (5.2).[6] For small λ, $\Sigma \sim \lambda^2 q^{2m}$ at $O(\partial^{2m})$, from figure 3. The extra powers of q in the denominator always stabilise the integral providing $\kappa > D/2 - 1$, but clearly the integral will then diverge as $\lambda \to 0$. A little

[6]Recall that these equations hold non-perturbatively, and it is still the case that the four-point functions in E need expanding as a power series in q.

further analysis taking into account the powers of λ in the four-point functions, shows that in the problematic cases of $O(\partial^{2m})$ with $m \geq \kappa + 3 - D/2$, the contribution behaves overall at small λ as $\sim \lambda^p$ where $p = -(2(m-k) + D - 4)/(m-1)$ is a negative fractional power. Obviously then, although the integrals converge, the results are qualitatively unacceptable.

Clearly these problems are generic, i.e., not just for contribution figure 2*b*.

7. Conclusions and outlook

We summarise each case in turn.

Using the Wilson/Polchinski equation, the one-loop β function calculation computed via a derivative expansion, results in a divergent numerical series for any cutoffs falling as a power or as the exponential of any power, in agreement with ref. [9]. Sharp cutoffs require the use of the Legendre flow equations or equivalent in any case. However, convergent numerical series are obtained for certain entire fast falling cutoffs such as $C_{\rm UV}(x) = \exp(1 - e^x)$, or $C_{\rm UV} = \exp\{e - \exp(e^x)\}$.

For the Legendre flow equations and any cutoff, the one-loop β function is exact within the derivative (or momentum-scale) expansion.

Derivative expansions of the Legendre flow equations using power law cutoffs $\sim 1/x^\kappa$, fail first at two loops, again as reported in ref. [9]. In fact here the $O(\partial^{2m})$ result, where $m \geq \kappa + 3 - D/2$, diverges. The reason is that certain contributions at these orders diverge for small coupling λ, and thus these $O(\partial^{2m})$ results are simply qualitatively incorrect in this regime. In $D = 4$ dimensions this problem first appears at $O(\partial^6)$.

The two-loop β function computed via a momentum-scale expansion using sharp cutoff, yields overall a slowly converging numerical series. The same diagrammatic contributions expanded to $O(k^2)$ in external momenta k result in series that fail to converge. These investigations improve on those reported in ref. [9] where we claimed on the basis of incomplete calculations that the two-loop β function yields a fast converging series [1, 10]. In fact as we have seen, all contributions converge very fast except for one, which turns out to be only very slowly converging. The analogous contribution to this, at $O(k^2)$, fails to converge at all.

Finally, the two-loop β function computed via a derivative expansion of the Legendre effective action and using an exponential cutoff $C_{\rm UV}(x) = e^{-x}$, yields a fast converging numerical series and moreover, the same diagrams expanded to any power of the external momenta give series that converge just as fast. These results verify and extend those reported in ref. [9].

Clearly we can conclude that sharp cutoffs and power-law cutoffs have inherent limitations, which is rather a shame since these are the only two cutoffs that continue to preserve reparametrization invariance, when combined with derivative/ momentum-scale expansions [7, 9].

JHEP08(1999)007

On the other hand, use of the Legendre flow equation and exponential cutoff— precisely as favoured by a number of authors [5, 19, 21], yields series that converge very well to two loops. Although our work has been limited at two loops to two-point and four-point vertices, it is natural to speculate that convergence is found for all operators at two loops. Certainly it is possible to investigate other, or maybe all, operators with the techniques used here. Although we concentrated on four dimensions, it is clear (from the discussion at the end of section 5), that convergence will be obtained in any dimension D. Naturally the question arises, what happens at three loops? These calculations are slightly more involved over and above the extra loop, because there are now two levels of embedding of subdiagrams, each needing expansion to a given order in the derivative expansion. Nevertheless these investigations again look possible and the results would be very interesting.

The use of $C_{\rm UV}(x) = e^{-x}$ is certainly very helpful in these investigations since the derivative expansion coefficients can be calculated exactly, as we have seen. Clearly the cutoff must fall faster than a power, but what else is required to obtain good results?

Although we have identified certain cutoffs to use with the Wilson/Polchinski equation that give convergent series for the one-loop β function, we did not investigate whether these series converge for higher derivative operators. This would seem to be an important and straightforward investigation. Another important question is the behaviour at two loops (and higher).

Clearly we leave many questions unanswered. (We have to stop somewhere!) Nevertheless it must be stressed again that the proof of convergence in the many examples we have worked out here, guarantee the accuracy of high orders of the derivative expansion for these quantities in the small coupling regime. With the derivative expansion known to be exact in various other limits (as reviewed in the introduction), these results further help to explain the impressive accuracy that may be obtained with these methods.

Acknowledgments

TRM acknowledges support of the SERC/PPARC through an Advanced Fellowship, and PPARC grant GR/K55738. JFT thanks PPARC for support through a studentship.

References

[1] T.R. Morris, *The exact renormalization group and approximate solutions*, *Int. J. Mod. Phys.* **A 9** (1994) 2411 [hep-ph/9308265].

[2] T.R. Morris, *Elements of the continuous renormalization group*, *Prog. Theor. Phys. Suppl.* **131** (1998) 395 [hep-th/9802039].

[3] J.F. Nicoll, T.S. Chang and H.E. Stanley, *Approximate renormalization group based on the Wegner-Houghton differential generator*, *Phys. Rev. Lett.* **33** (1974) 540.

[4] F.J. Wegner and A. Houghton, *Renormalization group equation for critical phenomena*, *Phys. Rev.* **A 8** (1973) 401;
M. D'Attanasio and T.R. Morris, *Large-N and the renormalization group*, *Phys. Lett.* **B 409** (1997) 363 [hep-th/9704094];
S.-K. Ma, *Introduction to the renormalization group*, *Rev. Mod. Phys.* **45** (1973) 589;
T.S. Chang, D.D. Vvedensky and J.F. Nicoll, *Differential renormalization-group generators for static and dynamical phenomena*, *Phys. Rev.* **217** (1992) 279;
K.-I. Aoki et al., *The effectiveness of the local potential approximation in the Wegner-Houghton renormalization group Prog. Theor. Phys.* **95** (1996) 409 [hep-ph/9612458].

[5] M. Reuter, N. Tetradis and C. Wetterich, *The large N limit and the high temperature phase transition for the ϕ^4 theory*, *Nucl. Phys.* **B 401** (1993) 567 [hep-ph/9301271].

[6] T.R. Morris and M. Turner, *Derivative expansion of the renormalization group in $O(N)$ scalar field theory*, *Nucl. Phys.* **B 509** (1998) 637 [hep-th/9704202].

[7] T.R. Morris, *Derivative expansion of the exact renormalization group*, *Phys. Lett.* **B 329** (1994) 241 [hep-ph/9403340].

[8] T.R. Morris, *Three-dimensional massive scalar field theory and the derivative expansion of the renormalization group*, *Nucl. Phys.* **B 495** (1997) 477 [hep-th/9612117].

[9] T.R. Morris, *Properties of derivative expansion approximations to the renormalization group Int. J. Mod. Phys.* **B 12** (1998) 1343 [hep-th/9610012].

[10] T.R. Morris, *Momentum scale expansion of sharp cutoff flow equations*, *Nucl. Phys.* **B 458** (1996) 477 [hep-th/9508017].

[11] T.R. Morris, in *New developments in the continuous renormalization group* in *New Developments in Quantum Field Theory*, NATO ASI series B, 366, Plenum Press 1998, hep-th/9709100.

[12] J.R. Shepard, V. Dmitrasinovic and J.A. McNeil, *Monte Carlo and renormalization group effective potentials in scalar field theories*, *Phys. Rev.* **D 51** (1995) 7017;
J.D. Shafer and J.R. Shepard, *Monte carlo and renormalization group local effective potentials in scalar field theories at finite temperature*, *Phys. Rev.* **D 55** (1997) 4990.

[13] R.D. Ball, P.E. Haagensen, J.I. Latorre, and E. Moreno, *Scheme independence and the exact renormalization group Phys. Lett.* **B 347** (1995) 80 [hep-th/9411122].

[14] G.R. Golner, *Nonperturbative renormalization-group calculations for continuum spin systems*, *Phys. Rev.* **B 33** (1986) 7863;
J. Comellas, *Polchinski equation, reparameterization invariance and the derivative expansion*, *Nucl. Phys.* **B 509** (1998) 662 [hep-th/9705129].

[15] G. Zumbach, *The renormalization group in the local potential approximation and its applications to the $O(N)$ model*, *Nucl. Phys.* **B 413** (1994) 754; *The local potential approximation of the renormalization group and its applications Phys. Lett.* **A 190** (1994) 225;
K.-I. Aoki et al., *The effectiveness of the local potential approximation in the wegner-houghton renormalization group*, *Prog. Theor. Phys.* **95** (1996) 409 [hep-ph/9612458];
J. Comellas and A. Travesset, *$O(N)$ models within the local potential approximation*, *Nucl. Phys.* **B 498** (1997) 539 [hep-th/9701028];
R. Neves, Y. Kubyshin and R. Potting, *Polchinski ERG equation and 2d scalar field theory*, hep-th/9811151.

[16] Y. Kubyshin, *Exact renormalization group approach in scalar and fermionic theories*, *Int. J. Mod. Phys.* **B 12** (1998) 1321 [hep-th/9702082].

[17] T.R. Morris, *The renormalization group and two-dimensional multicritical effective scalar field theory*, *Phys. Lett.* **B 345** (1995) 139 [hep-th/9410141].

[18] J.F. Nicoll and T.S. Chang, *An exact one particle irreducible renormalization group generator for critical phenomena*, *Phys. Lett.* **A 62** (1977) 287;
M. Bonini et al., *Perturbative renormalization and infrared finiteness in the Wilson renormalization group: the massless scalar case*, *Nucl. Phys.* **B 409** (1993) 441 [hep-th/9301114].

[19] C. Wetterich, *Exact evolution equation for the effective potential*, *Phys. Lett.* **B 301** (1993) 90.

[20] C. Wetterich, private communication.

[21] N. Tetradis and C. Wetterich, *Critical exponents from effective average action*, *Nucl. Phys.* **B 422** (1994) 541 [hep-ph/9308214]; *The high temperature phase transition for ϕ^4 theories*, *Nucl. Phys.* **B 398** (1993) 659;
J. Berges, N. Tetradis and C. Wetterich, *Coarse graining and first order phase transitions Phys. Lett.* **B 393** (1997) 387 [hep-ph/9610354]; *Critical equation of state from the average action Phys. Rev. Lett.* **77** (1996) 873 [hep-th/9507159];
D.U. Jungnickel and C. Wetterich, *Flow equations for phase transitions in statistical physics and QCD* hep-ph/9902316;
S. Seide and C. Wetterich, *Equation of state near the endpoint of the critical line* cond-mat/9806372;
J. Berges and C. Wetterich, *Equation of state and coarse grained free energy for matrix models*, *Nucl. Phys.* **B 487** (1997) 675 [hep-th/9609019];
S. Bornholdt et al., *Flow of the coarse grained free energy for crossover phenomena*, *Int. J. Mod. Phys.* **A 14** (1999) 899 [cond-mat/9603129]; *High temperature phase transition in two scalar theories Phys. Rev.* **D 53** (1996) 4552 [hep-ph/9503282]; *Coleman-Weinberg phase transition in two scalar models Phys. Lett.* **B 348** (1995) 89 [hep-th/9408132];

M. Grater and C. Wetterich, *Kosterlitz-Thouless phase transition in the two dimensional linear sigma model*, *Phys. Rev. Lett.* **75** (1995) 378 [hep-ph/9409459];
T. Papenbrock and C. Wetterich, *Two loop results from one loop computations and non perturbative solutions of exact evolution equations*, *Z. Physik* **C 65** (1995) 519 [hep-th/9403164];
D. Litim et al., *Nonperturbative analysis of the Coleman-Weinberg phase transition*, *Mod. Phys. Lett.* **A 12** (1997) 2287 [hep-ph/9407267];
N. Tetradis, *Comparison of renormalization group and lattice studies of the electroweak phase transition*, *Phys. Lett.* **B 409** (1997) 355 [hep-ph/9611461].

[22] M. Bonini et al., *Beta function and infrared renormalons in the exact Wilson renormalization group in Yang-Mills theory*, *Nucl. Phys.* **B 483** (1997) 475 [hep-th/9604114].

[23] K. Wilson and J. Kogut, *The renormalization group and the epsilon expansion*, *Phys. Rep.* **12** (1974) 75.

[24] J. Polchinski, *Renormalization and effective lagrangians Nucl. Phys.* **B 231** (1984) 269.

[25] e.g., J. Zinn-Justin *Quantum field theory and critical phenomena* C.P., Oxford 1993.